Linear algebra - Practice problems for midterm 2

1. Let $T : \mathcal{P}_2 \to \mathcal{P}_3$ be the linear transformation given by

$$T(p(x)) = \frac{dp(x)}{dx} - xp(x),$$

where $\mathcal{P}_2, \mathcal{P}_3$ are the spaces of polynomials of degrees at most 2 and 3 respectively.

(a) Find the matrix representative of T relative to the bases $\{1, x, x^2\}$ and $\{1, x, x^2, x^3\}$ for $\mathcal{P}_2$ and $\mathcal{P}_3$.

Solution: $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(b) Find the kernel of T.
Solution: The kernel is $\{0\}$.

(c) Find a basis for the range of T.
Solution: This is the same as the column space of the matrix in (a), but expressed as elements of $\mathcal{P}_3$. Row reducing the matrix we find that the range has basis $\{-x, 1 - x^2, 2x - x^3\}$.

2. Determine whether the following subsets of $\mathcal{P}_3$ are subspaces.

(a) $U = \{p(x) \ : \ p(3) = 0\}$
Solution: This is a subspace. If $p(x), q(x) \in U$, then $(p+q)(3) = p(3) + q(3) = 0$, so $p(x) + q(x) \in U$. Also $(rp)(3) = r \cdot p(3) = 0$, so $rp(x) \in U$ for any $r \in \mathbb{R}$.

(b) $V = \{p(x) \ : \ p(0) = 1\}$
Solution: This is not a subspace because it does not contain the zero polynomial.

(c) $W = \{p(x) \ : \ \text{the coefficient of } x^2 \text{ in } p(x) \text{ is } 0\}$.
Solution: This is a subspace, because if $p(x), q(x)$ have no x^2 term, then neither do $p(x) + q(x)$ and $rq(x)$ for $r \in \mathbb{R}$.

3. Let $M_{m\times n}$ be the vector space of $m \times n$ matrices, with the usual operations of addition and scalar multiplication.

(a) Let A be an $m \times m$ matrix. Is the function

$$T : M_{m\times n} \to M_{m\times n}$$

given by $T(B) = AB$ a linear transformation?
Solution: It is a linear transformation. We need to check $T(B+C) = T(B) + T(C)$ and $T(rB) = rT(B)$:

$$T(B+C) = A(B+C) = AB + AC = T(B) + T(C)$$
$$T(rB) = A(rB) = rAB = rT(B).$$

(b) Let $V \subset M_{m\times n}$ be the subset consisting of those matrices, whose entries all add up to zero. Is V a subspace of $M_{m\times n}$?
Solution: This is a subspace, since if A, B have entries adding up to zero, then so do $A + B$ and rA for any $r \in \mathbb{R}$.

4. Show that the subspaces $\mathrm{sp}(x - x^2, 2x)$ and $\mathrm{sp}(x^2, 3x + x^2)$ of $\mathcal{P}_2$ are equal.
Solution: We first show that $x - x^2, 2x \in \mathrm{sp}(x^2, 3x + x^2)$:

$$\begin{aligned} x - x^2 &= \frac{1}{3}(3x + x^2) - \frac{1}{3}x^2 - x^2 \\ 2x &= \frac{2}{3}\Big[(3x + x^2) - x^2\Big]. \end{aligned}$$

It follows that $\mathrm{sp}(x - x^2, 2x)$ is a subspace of $\mathrm{sp}(x^2, 3x + x^2)$. Similarly, from

$$\begin{aligned} x^2 &= -(x - x^2) + \frac{1}{2}(2x) \\ 3x + x^2 &= -(x - x^2) + \frac{1}{2}(2x) + \frac{3}{2}(2x), \end{aligned}$$

it follows that $\mathrm{sp}(x^2, 3x + x^2)$ is a subspace of $\mathrm{sp}(x - x^2, 2x)$. This means that the two subspaces are equal.

5. Find a basis for the subspace $\mathrm{sp}(1 + x^2, 2x - x^2, 4x + 2)$ of $\mathcal{P}_3$.
Solution: Using the basis $\{1, x, x^2, x^3\}$ for $\mathcal{P}_3$, we can write the vectors $1 + x^2, 2x - x^2, 4x + 2$ as the columns of the matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Row reducing the matrix, we find that a basis is given by $\{1 + x^2, 2x - x^2\}$.

6. Working in the space $\mathcal{P}_3$, find the coordinate vector of x^2, relative to the basis $\{1, x - 1, (x - 1)^2, (x - 1)^3\}$.
Solution: We need to find a_1, a_2, a_3, a_4 such that

$$x^2 = a_1 + a_2(x - 1) + a_3(x - 1)^2 + a_4(x - 1)^3.$$

One can write this as a matrix, and we find $a_1 = 1, a_2 = 2, a_3 = 1, a_4 = 0$, so the coordinate vector of x^2 in the basis is $[1, 2, 1, 0]$.

7. Compute the determinant

$$\det \begin{bmatrix} 3 & -2 & 7 & 6 \\ -4 & 0 & 2 & 1 \\ 5 & 2 & 0 & -2 \\ 2 & 0 & -1 & 0 \end{bmatrix}$$

Solution: Use column and row operations to simplify the calculation:

$$\begin{aligned} \det \begin{bmatrix} 3 & -2 & 7 & 6 \\ -4 & 0 & 2 & 1 \\ 5 & 2 & 0 & -2 \\ 2 & 0 & -1 & 0 \end{bmatrix} &= -\det \begin{bmatrix} -2 & 3 & 7 & 6 \\ 0 & -4 & 2 & 1 \\ 2 & 5 & 0 & -2 \\ 0 & 2 & -1 & 0 \end{bmatrix} \\ = -\det \begin{bmatrix} -2 & 3 & 7 & 6 \\ 0 & -4 & 2 & 1 \\ 0 & 8 & 7 & 4 \\ 0 & 2 & -1 & 0 \end{bmatrix} &= \det \begin{bmatrix} -2 & 3 & 7 & 6 \\ 0 & 2 & -1 & 0 \\ 0 & 8 & 7 & 4 \\ 0 & -4 & 2 & 1 \end{bmatrix} \\ = \det \begin{bmatrix} -2 & 3 & 7 & 6 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 11 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} &= -44 \end{aligned}$$

8. Suppose that A is an $n \times n$ matrix, such that all of the entries of A add up to zero. Is it true that $\det(A) = 0$?
Solution: It is not true, for example

$$\det \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = 1.$$